

The role of phases in detecting three qubit entanglement*

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Abstract

We propose separability criteria for three qubit states in terms of diagonal and anti-diagonal entries to detect entanglement with positive partial transposes. We report here that the phases of anti-diagonal entries play a crucial role. In some cases, the anti-diagonal phases of separable states must satisfy even an identity. These criteria are strong enough to detect PPT entanglement with nonzero volume. Our criteria give us complete characterizations of separability when entries of a state are zero except for diagonal and anti-diagonals, with a common magnitude for anti-diagonals.

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The notion of entanglement is a unique phenomenon in quantum physics, and is now considered as one of the main resources in various fields of current quantum information and computation theory, like quantum cryptography and quantum teleportation. See survey articles [1, 2] for general aspects on the topics. In cases of multi-partite systems, there are several kinds of entanglement as it was classified in [3–5], and it is important to find separability criteria to distinguish entanglement from separability. Positivity of partial transposes is a simple but powerful criterion [6, 7].

Some of other criteria are to test quite simple relations between diagonal and anti-diagonal entries of states. See [8–10] for example. This approach is very successful to detect kinds of multi-qubit entanglement arising from bi-separability or full bi-separability, and some of them actually characterize those separability when a given state has zero entries except for diagonal and anti-diagonal entries [8, 10, 11]. We recall that a state is said to be separable, or fully separable, if it is a mixture of pure product states, or equivalently, that of product states. A multi-partite state can be interpreted as a bi-partite state if we give a bi-partition for the systems. A multi-partite state is called an $(S|T)$ bi-separable if it is separable as a bi-partite state according to the bi-partition $(S|T)$ for systems. It is called fully bi-separable if it is $(S|T)$ bi-separable for every bi-partition $(S|T)$, and bi-separable if it is a mixture of $(S|T)$ bi-separable states through bi-partitions $(S|T)$ for systems. In general, a state is called entangled if it is not separable. Therefore, we have several notions of separability and entanglement.

States with zero entries except for diagonal and anti-diagonal entries are usually called X-shaped states, or X-states in short. Those states arise naturally in quantum information theory in various aspects. See [12–16] for example. Notable examples include Greenberger-Horne-Zeilinger diagonal states, which are mixtures of GHZ states with noises. We note that the X-part of a three qubit separable state is again separable [17], and so any necessary criteria for separability of X-shaped states still work for arbitrary three qubit states in terms of diagonal and anti-diagonal entries. Separability of three qubit X-states is still vague, even though we now completely understand bi-separability and full bi-separability of arbitrary multi-qubit X-states as mentioned above. Only recently, separability of three qubit GHZ diagonal states has been completely characterized by the authors [17], complimenting earlier partial results in [9, 18]. We note that anti-diagonal entries of GHZ diagonal states are real numbers, and so it is very natural to ask what happens when the anti-diagonal part has

complex entries.

In this article, we provide separability criteria in terms of diagonal and anti-diagonal entries to detect three qubit entanglement, which depends on phases of anti-diagonal entries. Anti-diagonal phases play a role in general to determine positivity of Hermitian matrices. For example, if we consider the $n \times n$ matrix $[a_{i,j}]$ whose entries are all 1 except for $a_{1,n} = a_{n,1}^* = e^{i\theta}$ with $n \geq 3$, then it is positive only when $\theta = 0$. But, they play no role for positivity of X-shaped Hermitian matrices. This means that criterion for positivity with diagonal and anti-diagonal entries depends only on the magnitudes of entries. This is also the case for bi-separability of multi-qubit states. In fact, full bi-separability and bi-separability of multi-qubit X-states are equivalent to the corresponding notions of positivity of partial transposes [11]. Nevertheless, we report here that phases of anti-diagonal entries play a crucial role to determine (full) separability of three qubit X-states. In other words, detecting entanglement with the PPT property depends on the anti-diagonal phases. Furthermore, PPT entanglement detected in this way has nonzero volume. We have already seen [17, 19] the role of anti-diagonal phases to determine if a given X-shaped three qubit Hermitian matrix is an entanglement witness or not. Therefore, it is reasonable to expect corresponding roles of anti-diagonal phases in the separability criteria.

Three qubit states will be considered as 8×8 matrices, and so, a three qubit X-shaped self-adjoint matrix is of the form

$$X(a, b, c) = \begin{pmatrix} a_1 & & & & & & & c_1 \\ & a_2 & & & & & & c_2 \\ & & a_3 & & & & & c_3 \\ & & & a_4 & c_4 & & & \\ & & & \bar{c}_4 & b_4 & & & \\ & & & & & \bar{c}_3 & b_3 & \\ & & & & & \bar{c}_2 & b_2 & \\ & & & & & & \bar{c}_1 & b_1 \end{pmatrix},$$

for $a, b \in \mathbb{R}^4$ and $c \in \mathbb{C}^4$. We also denote by θ_i the phase of $c_i = r_i e^{i\theta_i}$. The main purpose of this article is to show that the separability of a state ϱ with the X-part $X(a, b, c)$ depends on the *phase difference* defined by

$$\phi_\varrho = \frac{1}{2}(\theta_1 + \theta_4) - \frac{1}{2}(\theta_2 + \theta_3).$$

We recall that if a three qubit state ϱ with the X-part $X(a, b, c)$ is separable then it satisfies the inequality

$$\Delta_\varrho \geq R_\varrho, \quad (1)$$

where Δ_ϱ is the minimum of the six numbers $\sqrt{a_i b_i}$ for $i = 1, 2, 3, 4$, $\sqrt[4]{a_1 b_2 b_3 a_4}$ and $\sqrt[4]{b_1 a_2 a_3 b_4}$, given by the diagonal entries, and R_ϱ is the maximum of $|c_i|$. The inequalities $\sqrt{a_i b_i} \geq R_\varrho$ come from the positivity of partial transposes, as it was also observed for general multi-qubit states [11]. The other two inequalities appear in [9].

The criterion (1) gives us a restriction on the maximum of magnitudes of anti-diagonal entries for separable states. We consider here the minimum r_ϱ of the anti-diagonal magnitudes, and show that a separable state ϱ must satisfy the inequality:

$$\Delta_\varrho \geq r_\varrho \sqrt{1 + |\sin \phi_\varrho|}, \quad (2)$$

which depends on the phase difference ϕ_ϱ . Note that the number $|\sin \phi_\varrho|$ in the criterion is invariant under three kinds of partial transposes of ϱ . If the phase difference is nonzero then our criterion (2) detects entanglement with PPT property. For given fixed diagonal parts and magnitudes of anti-diagonals, our criterion also gives a restriction on the phase difference for separable states. Especially, if $\Delta_\varrho = R_\varrho = r_\varrho$, then separable states must obey the *phase identity*: $\theta_1 + \theta_4 = \theta_2 + \theta_3$, which can be easily observed for pure product states [9].

In order to compare two criteria (1) and (2), we fix the diagonal part a, b and the *phase part* $(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})$, and consider the four dimensional convex body \mathbb{S} consisting of (r_1, r_2, r_3, r_4) so that the X-state $X(a, b, c)$ is separable. The criterion (1) tells us that \mathbb{S} is sitting in the cube with the width Δ_ϱ . On the other hand, we see that the region \mathbb{S} is located in the union of strips with width $\Delta_\varrho / \sqrt{1 + |\sin \phi_\varrho|}$ by (2). See FIGURE 1. We will also show that the criterion (2) gives rise to a complete characterization of separability when ϱ is an X-state with a common magnitude for anti-diagonals. This tells us that the ‘corner point’ of the strips belongs to the region \mathbb{S} , and so, the bound (2) for the minimum of magnitudes is optimal. Furthermore, the ‘corner point’ represents a boundary separable state whose partial transposes are of full ranks, whenever $\phi_\varrho \neq 0$. Construction of such states has been asked in [20] and answered in [21]. We add here more examples. If we take an X-state ϱ satisfying the strict inequality in (1) but violating (2), then ϱ is a PPT entanglement, and all of its partial transposes have full ranks. Therefore, ϱ is an interior point of the set of all

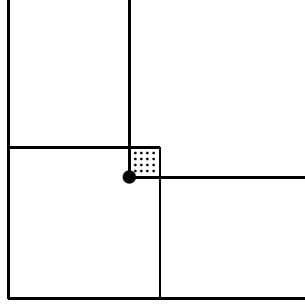


FIG. 1. The square in the picture represents the four dimensional cube determined by the criterion (1), and the union of strips is the region determined by (2). So, the region \mathbb{S} located in the intersection of these regions, and the dotted part is PPT entanglement detected by the criterion (2). The ‘corner point’ of the strip, shown by a big dot, is on the boundary of the separability region \mathbb{S} .

PPT states, and so we see that PPT entanglement detected by (2) has nonzero volume. In general cases with arbitrary anti-diagonal magnitudes, we show that the inequality

$$\Delta_\varrho \geq R_\varrho \sqrt{1 + \max\{|\sin \phi_\varrho|, |\cos \phi_\varrho|\}} \quad (3)$$

gives rise to a sufficient condition for separability of X-states. Therefore, the region \mathbb{S} lies between two cubes determined by (1) and (3).

In order to justify the criteria (2), we begin with the separability criterion by Gühne [9] which was simplified by the authors [17]. It was shown that every three qubit separable state ϱ with the X-part $X(a, b, c)$ satisfies the inequality

$$\begin{aligned} & |\operatorname{Re}(z_1 c_1 + z_2 c_2 + z_3 c_3 + z_4 \bar{c}_4)| \\ & \leq \Delta_\varrho \max_{\tau} (|z_1 e^{i\tau} + z_4| + |z_2 e^{i\tau} + \bar{z}_3|) \end{aligned} \quad (4)$$

for each $z \in \mathbb{C}^4$. Taking $z = (e^{i(\theta+\phi)}, e^{i\phi}, e^{i(\theta+\psi)}, -e^{-i\psi})$ with $\phi = -\arg(c_1 e^{i\theta} + c_2)$ and $\psi = -\arg(c_3 e^{i\theta} - c_4)$ for a given θ , we see that the left side becomes $|c_1 e^{i\theta} + c_2| + |c_3 e^{i\theta} - c_4|$. Furthermore, the maximum of the right side is $2\sqrt{2}$ through the variable τ . Considering the number

$$A_\varrho = \frac{1}{2\sqrt{2}} \max_{\theta} (|c_1 e^{i\theta} + c_2| + |c_3 e^{i\theta} - c_4|)$$

which is also determined by anti-diagonal parts, we have the separability criterion

$$\Delta_\varrho \geq A_\varrho. \quad (5)$$

Note that the number A_ϱ is invariant under all the kinds of partial transposes.

Now, consider the case when the anti-diagonal entries share a common magnitude, say $R = |c_i|$ for each $i = 1, 2, 3, 4$. In this case, the number A_ϱ has a natural geometric interpretation. To see this, put

$$\begin{aligned} T(\theta) &:= |e^{i\theta_1}e^{i\theta} + e^{i\theta_2}| + |e^{i\theta_3}e^{i\theta} - e^{i\theta_4}| \\ &= |e^{i\theta} - e^{i(\theta_2 - \theta_1 + \pi)}| + |e^{i\theta} - e^{i(\theta_4 - \theta_3)}|. \end{aligned}$$

Then the maximum of $T(\theta)$ occurs when the three points $e^{i(\theta_2 - \theta_1 + \pi)}$, $e^{i(\theta_4 - \theta_3)}$ and $e^{i\theta}$ make an isosceles triangle on the complex plane, and so we have $2\sqrt{2} \leq \max_\theta T(\theta) \leq 4$. See FIGURE 2. Since $A_\varrho = R \max_\theta T(\theta)/2\sqrt{2}$, we have $R \leq A_\varrho \leq \sqrt{2} R$. Note that $A_\varrho = R$ if and only if $\max_\theta T(\theta) = 2\sqrt{2}$ if and only if two angles $\theta_2 - \theta_1 + \pi$ and $\theta_4 - \theta_3$ are antipodal if and only if $\theta_1 + \theta_4 = \theta_2 + \theta_3$. We also have $A_\varrho = \sqrt{2} R$ if and only if $\theta_1 + \theta_4$ and $\theta_2 + \theta_3$ are antipodal. Now, we have

$$\begin{aligned} \max_\theta T(\theta) &= \max_\theta T(\theta - \theta_3 + \theta_4) \\ &= \max_\theta \{|e^{i\theta} - e^{i(-2\phi_\varrho + \pi)}| + |e^{i\theta} - 1|\}, \end{aligned}$$

and its maximum occurs when three points 1 , $e^{i(-2\phi_\varrho + \pi)}$ and $e^{i\theta}$ make an isosceles triangle, that is, $\theta = -\phi_\varrho \pm \frac{\pi}{2}$. Therefore, we see that $\max_\theta T(\theta)$ is given by $\max_\pm 2|e^{i(-\phi_\varrho \pm \pi/2)} - 1|$. This shows the identity

$$A_\varrho = R \sqrt{1 + |\sin \phi_\varrho|}, \quad (6)$$

whenever $R = |c_i|$ for each $i = 1, 2, 3, 4$. In the general case with arbitrary magnitudes of anti-diagonal entries, we have

$$A_\varrho \geq r_\varrho \sqrt{1 + |\sin \phi_\varrho|}, \quad (7)$$

by the inequality $|c_1 e^{i\theta} \pm c_2| \geq \min\{r_1, r_2\} |e^{i\theta_1} e^{i\theta} \pm e^{i\theta_2}|$. By (5), we have the criterion (2).

We turn our attention to search for sufficient criteria for separability of an X-state $\varrho = X(a, b, c)$. Suppose that $\varrho = X(a, b, c)$ is a non-diagonal separable X-state of rank four. It is easily seen that the separability of ϱ implies the condition

$$a_1 a_4 = a_2 a_3, \quad \sqrt{a_i b_i} = |c_j|, \quad c_1 c_4 = c_2 c_3, \quad (8)$$

for each $i, j = 1, 2, 3, 4$. In fact, the first two conditions are equivalent to (1) in this case. Especially, this shows that the anti-diagonal entries share a common magnitude with $\Delta_\varrho =$

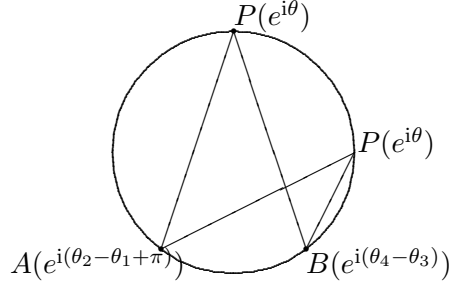


FIG. 2. For two fix points A and B on the circle, the length $\overline{AP} + \overline{PB}$ takes the maximum when the three points A, B and P make an isosceles triangle. Furthermore, the number $\max_P(\overline{AP} + \overline{PB})$ becomes largest when A and B coincide, and smallest when A and B are antipodal.

$R_\varrho = r_\varrho$, and so the last condition must hold by the phase identity. Conversely, we check that a non-diagonal rank four X-state $\varrho = X(a, b, c)$ with the condition (8) has exactly four product vectors in the range space whose average is just ϱ . Therefore, ϱ is a separable state with a unique decomposition. We note that generic choices of four product vectors in the three qubit system give rise to separable states with unique decomposition [22]. It is also easy to see that a three qubit X-state $\varrho = X(a, b, c)$ is separable if

$$\Delta_\varrho \geq R_\varrho = r_\varrho, \quad c_1 c_4 = c_2 c_3. \quad (9)$$

In fact, ϱ can be expressed as the sum of a rank four state with the condition (8) and a diagonal state in this case.

Now, we consider the case $R_\varrho = r_\varrho$, and show that the inequality (2) is sufficient for separability of $\varrho = X(a, b, c)$. Actually, we express ϱ as the mixture of two X-states with the condition (9). We assume that $|c_i| = 1$ and $0 \leq \phi_\varrho < \pi$ without loss of generality. For notational convenience, put $\phi = \phi_\varrho$, $r = A_\varrho = \sqrt{1 + |\sin \phi|}$. By the equality $r = \sin(\phi/2) + \cos(\phi/2)$, we have

$$\tan \frac{\phi}{2} (1 - r e^{i(\phi/2 - \pi/2)}) = r e^{i(\phi/2)} - 1,$$

which tells us that the three points $r e^{i(\phi/2)}, 1, r e^{i(\phi/2 - \pi/2)}$ are co-linear on the complex plane. Take nonnegative numbers p and q such that $p + q = 1$ and $1 = p r e^{i(\phi/2)} + q r e^{i(\phi/2 - \pi/2)}$. If we put $u = r e^{i(\phi/2)}$ and $v = r e^{i(\phi/2 - \pi/2)}$, then we have $pu + qv = 1$. Put $\varrho_1 = X(a, b, c')$ and $\varrho_2 = X(a, b, c'')$ with $c' = (c_1 \bar{u}, c_2 u, c_3 u, c_4 \bar{u})$ and $c'' = (c_1 \bar{v}, c_2 v, c_3 v, c_4 \bar{v})$. Then we have

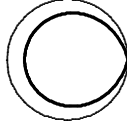


FIG. 3. The thick curve is $1 = r\sqrt{1 + |\sin(\theta/2)|}$ on the complex plane with the polar coordinate, which represents the boundary of the region for separability of the X-state $X(\mathbf{1}, \mathbf{1}, (r, r, re^{i\theta}, r))$. The circle represents the region of PPT property.

$\varrho = p\varrho_1 + q\varrho_2$. If the inequality (5) holds then it is easily checked that ϱ_1 and ϱ_2 satisfy the relation (9), and so they are separable. Therefore, the inequality (5) implies the separability of ϱ whenever $R_\varrho = r_\varrho$. This shows that (2) is also sufficient for separability in this case by (6).

Consider the X-shaped matrix $\varrho = X(\mathbf{1}, \mathbf{1}, c)$ with $\mathbf{1} = (1, 1, 1, 1)$ and $c = r(1, 1, e^{i\theta}, 1)$. Then ϱ is a state if and only if it is a PPT state if and only if $r \leq 1$. On the other hand, our criteria tell us that ϱ is separable if and only if $1 \geq r\sqrt{1 + |\sin(\theta/2)|}$. See FIGURE 3. In the special case of $c = (1, 1, -1, 1)$, we see that ϱ is separable if and only if $r \leq \frac{1}{\sqrt{2}}$, as it was shown in [9, 18]. This example shows clearly the role of phases for the criteria of separability.

Now, we show that (3) is sufficient for separability of an X-state $\varrho = X(a, b, c)$. For a string $\epsilon = \epsilon_1\epsilon_2\epsilon_3\epsilon_4$ of ± 1 , we consider the X-state $\varrho^\epsilon = (a, b, R_\varrho\Phi_\varrho^\epsilon)$ with the phase part $\Phi_\varrho^\epsilon = (\epsilon_1e^{i\theta_1}, \epsilon_2e^{i\theta_2}, \epsilon_3e^{i\theta_3}, \epsilon_4e^{i\theta_4})$. Since $|\sin \phi_{\varrho^\epsilon}| \leq \max\{|\sin \phi_\varrho|, |\cos \phi_\varrho|\}$ in general, the inequality (3) tells us that the criterion (2) holds for the state ϱ^ϵ . Because each ϱ^ϵ shares a common anti-diagonal magnitude R_ϱ , we see that ϱ^ϵ is separable for each string ϵ . This shows that the state ϱ is also separable since ϱ is a convex combination of ϱ^ϵ 's.

We examine various criteria for the X-shaped matrix $\varrho = X(\mathbf{1}, \mathbf{1}, (p, p, q, -q))$ with real p and q . Note that ϱ is a state if and only if it is of PPT if and only if $1 \geq \max\{|p|, |q|\}$. This is a GHZ diagonal state. The inequality (5) is just $1 \geq (|p| + |q|)/\sqrt{2}$, which does not detect entanglement when $|p|/|q|$ is big or small enough. One can also check that ϱ is separable if and only if $1 \geq \sqrt{p^2 + q^2}$ by the result in [17]. See FIGURE 4.

In this article, we got two separability criteria (2) and (5) for three qubit states in terms of diagonal and anti-diagonal entries. The criterion (2) shows the role of phases more directly, but it is weaker criterion than (5) by the inequality (7). These two criteria are equivalent

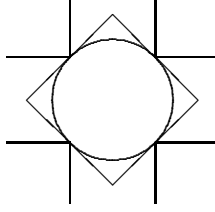


FIG. 4. The circle centered at the origin on the pq -plane represents the region for separability of $\varrho = X(\mathbf{1}, \mathbf{1}, (p, p, q, -q))$. The diamond and the union of horizontal and vertical strips represent the regions satisfying the inequalities (5) and (2), respectively. Two cubes by the conditions (1) and (3) are squares (which are not shown in the picture) circumscribing and inscribing the circle, respectively.

to each other when the anti-diagonal entries share a common magnitude. We note that $r_\varrho \leq R_\varrho \leq \Delta_\varrho$ for PPT states. Therefore, the smaller is the ratio R_ϱ/r_ϱ , the sharper is the criterion (2). In fact, we have shown that each of two criteria also gives rise to a sufficient condition for separability of X-states when $R_\varrho = r_\varrho$. On the other hand, they are of little use to detect entanglement when the ratio R_ϱ/r_ϱ is big, as we have seen in the example $X(\mathbf{1}, \mathbf{1}, (p, p, q, -q))$.

We note that these two criteria depend on the criterion (4) which is not so easy to apply directly. In the case that all the z_i 's are real, the maximum part in the criterion can be evaluated in terms of z_i 's. This was useful in [17] to characterize separability of GHZ diagonal states, where all the anti-diagonal entries are real numbers. It would be interesting to characterize separability of general three qubit X-states. It would be also nice to find multi-qubit analogues for three qubit phase differences.

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